Chapter 3: Temperature Estimates for Stars

3-1: Black Body

3-1-1: What is Black Body?

- **Thermodynamic equilibrium: the macroscopic thermodynamic quantities, such as temperature and pressure, are constants and unchanged with the time.**

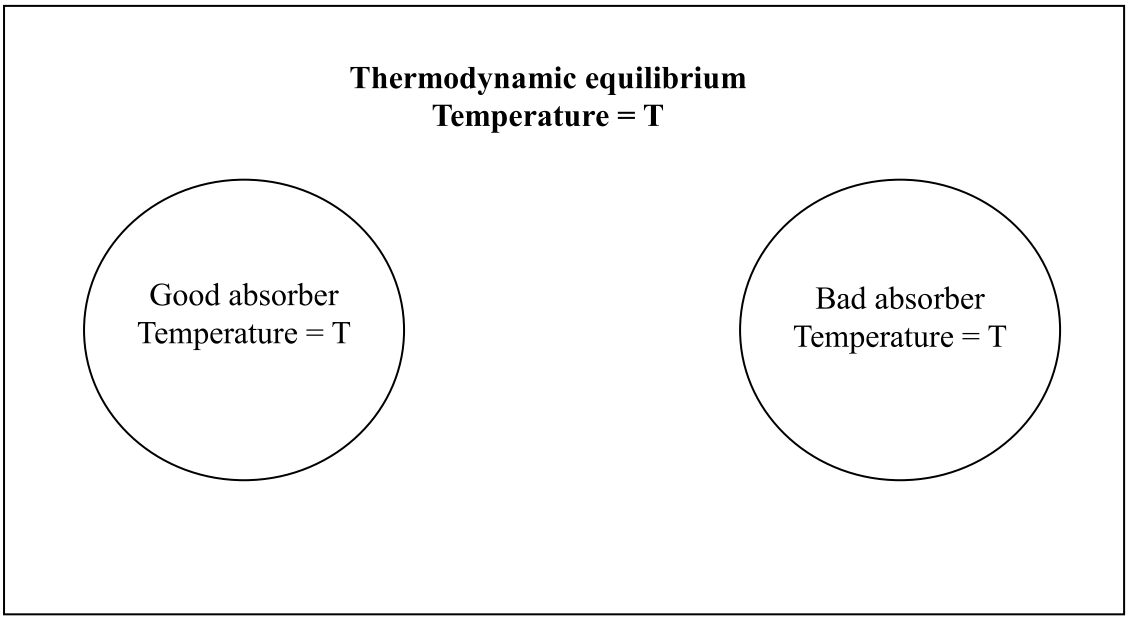
- A system keeps thermodynamic equilibrium with environment:
  $\left|\Delta Q_{in}\right| = \left|\Delta Q_{out}\right|$
  $\left|\Delta Q_{in}\right|$: heat into the system (from environment)
  $\left|\Delta Q_{out}\right|$: heat out of the system (into environment)

- Radiation: one of the three ways to transfer the heat.

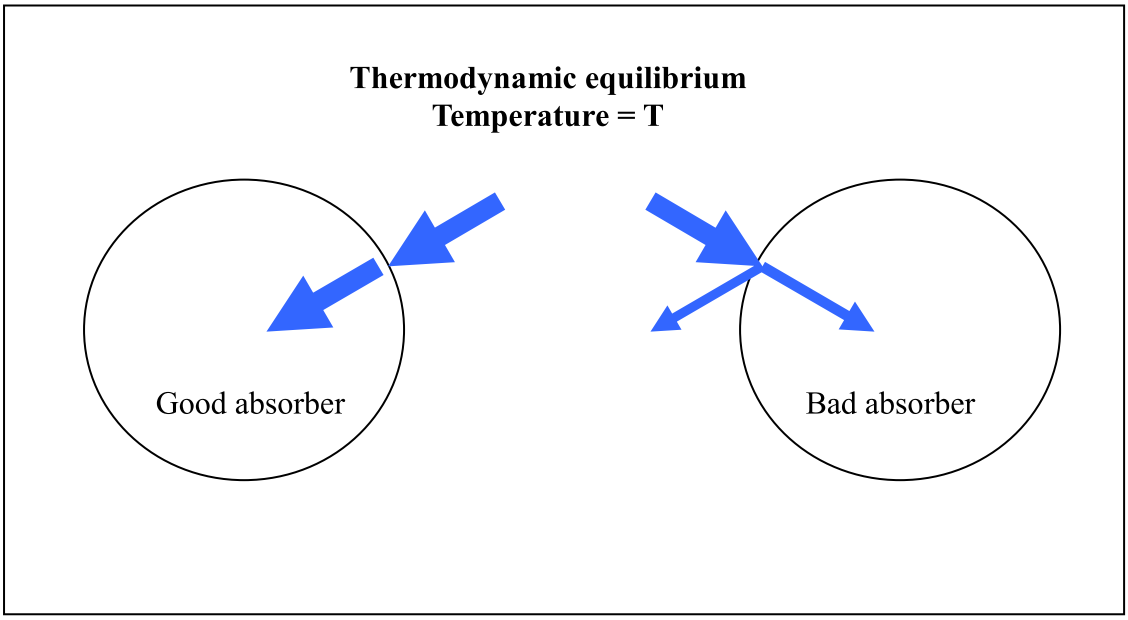
- **All materials would radiate electromagnetic wave if $T \neq 0$ and independent of the environment.**

- **Principle: a good absorber must be a good emitter and vise versa.**

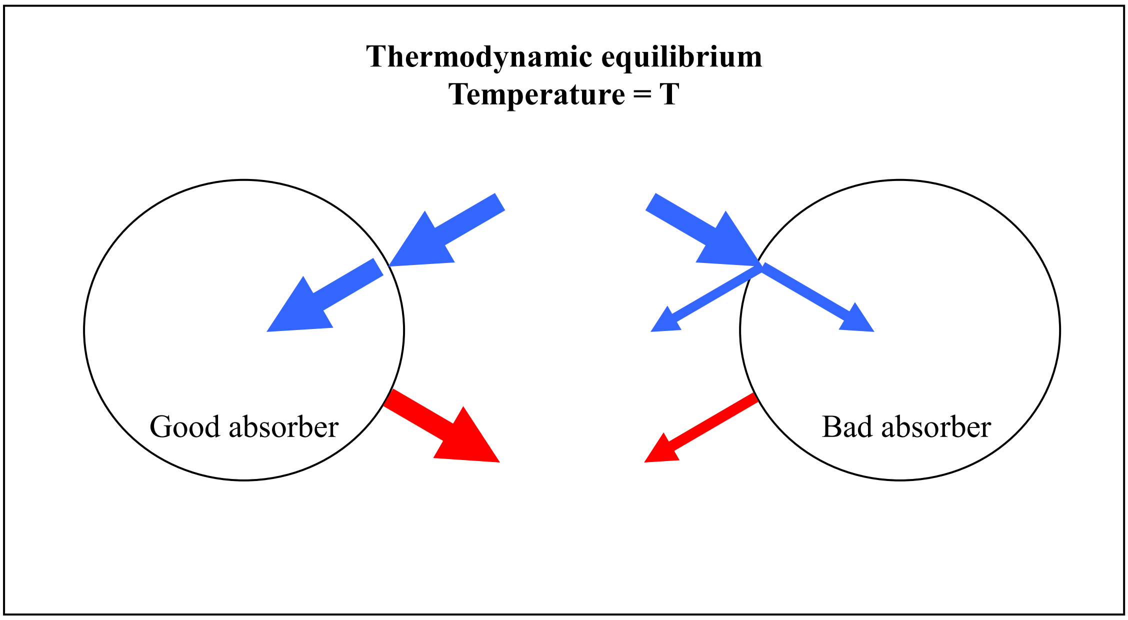
- Proof:
  - A vacuum box containing both a good absorber and a bad absorber with identical feature reaches thermodynamic equilibrium inside the box and thus:



  - The radiation absorbed by the good absorber is larger than the bad one.



■ However, since the system reaches thermodynamic equilibrium, the energy radiated out of good absorber must be equal to the energy absorbed by it and so does the bad one.



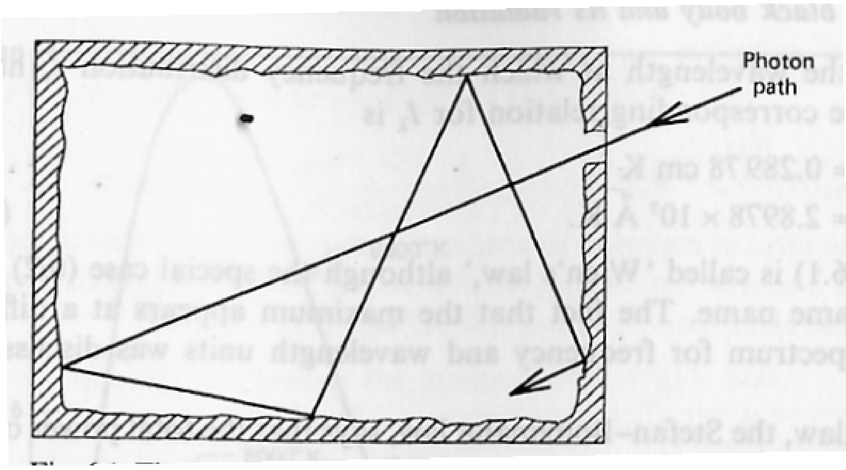
■ Since the radiation from the object is independent of the environment. We therefore prove that the good absorber must be a good emitter and vise versa.

- **The perfect absorber absorbs all the radiations incident into it** so it should looks "black" so it was named as "black body".

- **However, it doe not mean that the black body does not radiate.** The "black" here only means it absorbs all the radiations incident into it

3-1-2 Kirchhoff's Law(s)

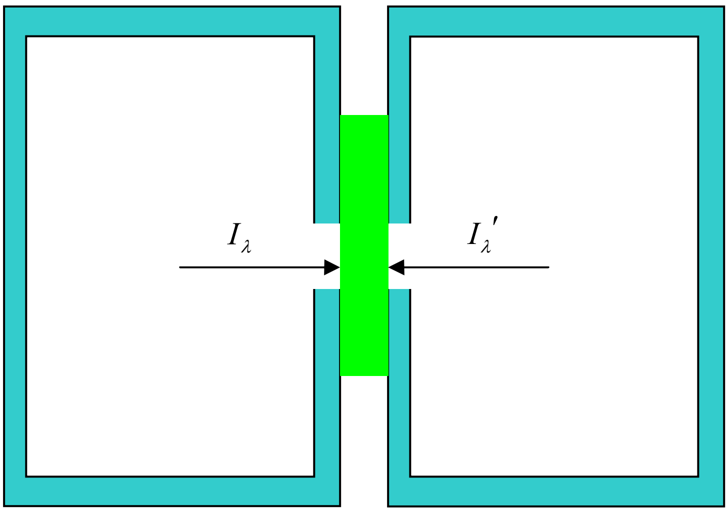
- **A (infinitesimal) hole on a box acts like a black body.** Any light entering the box will have a very small probability to escape and will eventually be absorbed by the gas or walls. For constant temperature walls, this is in thermodynamic equilibrium.



- If this box is heated the walls will emit photons, filling the inside with radiation. A small fraction of the radiation will leak out of the hole, but so little that the gas within it remains in thermodynamic equilibrium. The emitted radiation is that of a black body..

- **The emission spectrum of a blackbody is only a universal function of temperature:**
  - Proof: for two enclosures both temperature of T connected by the filter which allows only the radiation of certain wavelength λ passing through it, if the fluxes emitted from the two are different, there will be net heat flowing from one enclosure to the other, which is against the second law of thermodynamics.

    Thus $I_\lambda = I_\lambda'$
  - Thus, $I_\lambda = B_\lambda(T)$ = a universal function of temperature and λ



- **For the *thermodynamic equilibrium* in the box, no net energy can be added or removed from the rays**:

$$
\begin{aligned}
&\frac{dI_\lambda}{ds} = 0 \\
&\Rightarrow \kappa_\lambda I_\lambda = j_\lambda \\
&\Rightarrow j_\lambda = \kappa_\lambda B_\lambda \\
&\text{or} \\
&\frac{j_\lambda}{\kappa_\lambda} = B_\lambda(T) = S_\lambda
\end{aligned}
$$

- **However, the source function is the property of the medium regarding the spontaneous specific emission and absorption coefficients of the medium, which is independent of the radiation field. Thus, at temperature T, the source function**

$$\frac{j_\lambda}{\kappa_\lambda} = B_\lambda(T) = S_\lambda(T)$$

- This is the Kirchhoff's law of radiation.

- Thus, in general

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + B_\lambda(T)$$

- The cavity (box) case discussed above is equivalent to the optical thick case.

- **Thermal radiation:** $S_\lambda = B_\lambda$

- **Black body radiation:** $I_\lambda = B_\lambda$

- **The thermal radiation becomes black body only for optically thick.**

- **Black body radiation is isotropic.**

- There are other Kirchhoff's radiation laws

  - **First law:. A hot dense object produces light with a continuous spectrum**
    - The "dense" here can be treated as the object is optically thick.
    - If the object is in thermal equilibrium, the emission is black body.

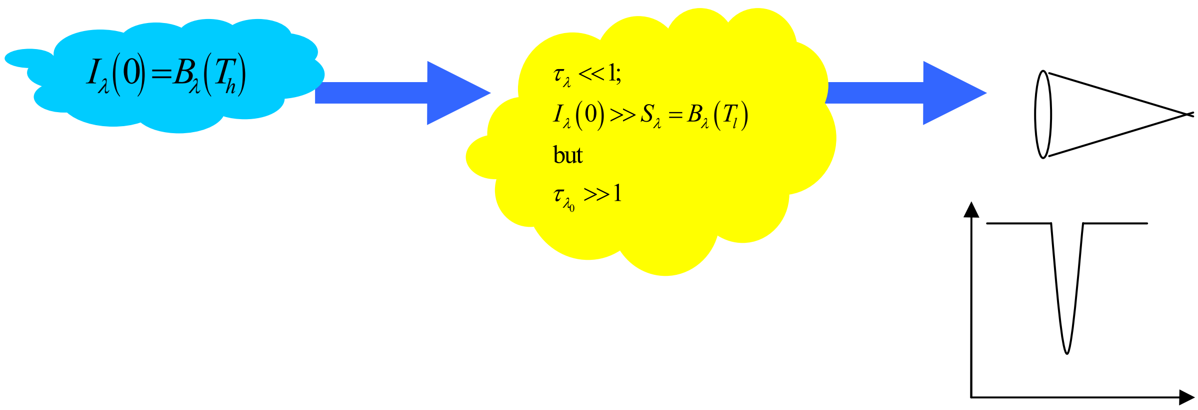
  - **Second law: A hot tenuous gas produces emission spectral lines at discrete wavelengths which depend on the energy levels of the atoms in the gas.**
    - The "tenuous" here means optically thin.
    - The "hot" here means the source function $S_\lambda = B_\lambda$ not small.
    - This is the case $I_\lambda(0) = 0; \tau_\lambda << 1$ but $\tau_{\lambda_0} >> 1$ in Chapter 2, thus

$I_\lambda = \tau_\lambda B_\lambda(T)$ is small

but

$I_{\lambda_0} = B_\lambda(T)$ is not small

- **Third law: A hot dense object surrounded by a cool tenuous gas (i.e. cooler than the hot object) produces light with an almost continuous spectrum which has gaps at discrete wavelengths depending on the energy levels of the atoms in the gas**
  - This is the case $B_\lambda(T_h) = I_\lambda(0) >> S_\lambda = B_\lambda(T_l), \quad \tau_\lambda << 1$



3-2 Black Body Radiation

3-2-1 Stefan-Boltzmann Law

- **The flux energy density of the vacuum box**

$$u_\lambda = \frac{1}{c}\int I_\lambda d\Omega = \frac{B_\lambda(T)}{c}\int d\Omega = \frac{4\pi B_\lambda(T)}{c}$$

$$u(T) = \int u_\lambda(T)\, d\lambda$$

Thus the energy density is only a function of temperature.

- **The total energy**

$$U(T,V) = u(T)V$$

where V is the volume of the box

- **The pressure from the radiation** in the box can be written as

$$P = \frac{u}{3}$$

- **Use the thermodynamic identity for entropy** S

$$dS = \left(\frac{\partial S}{\partial U}\right)_V dU + \left(\frac{\partial S}{\partial V}\right)_U dV$$

$$= \frac{1}{T}dU + \frac{P}{T}dV$$

$$= \frac{1}{T}dU + \frac{1}{3}\frac{u}{T}dV$$

Since

$$U = U(T,V) = u(T)V$$

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

$$= V\frac{du}{dT}dT + u dV$$

$$\therefore dS = \frac{1}{T}\left(V\frac{du}{dT}dT + u dV\right) + \frac{1}{3}\frac{u}{T}dV$$

$$= \frac{V}{T}\frac{du}{dT}dT + \frac{4}{3}\frac{u}{T}dV$$

- Since the entropy is perfect differential

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{V}{T}\frac{du}{dT}; \quad \left(\frac{\partial S}{\partial V}\right)_T = \frac{4}{3}\frac{u}{T}$$

$$\therefore \frac{\partial S}{\partial V \partial T} = \frac{1}{T}\frac{du}{dT} = \frac{\partial S}{\partial T \partial V} = -\frac{4}{3}\frac{u}{T^2} + \frac{4}{3}\frac{1}{T}\frac{du}{dT}$$

$$\Rightarrow \frac{du}{dT} = 4\frac{u}{T}$$

$$\Rightarrow \frac{du}{u} = 4\frac{dT}{T}$$

$$\Rightarrow \ln u = 4\ln T + \ln a$$

where a is a constant

$$u = aT^4$$

- **Thus, the brightness**

$$u = \frac{4\pi}{c}\int B_\lambda(T)\,d\nu = \frac{4\pi}{c}B(T)$$

$$\therefore B(T) = \frac{c}{4\pi}u = \frac{ca}{4\pi}T^4$$

- **Thus the flux on the surface is**

$$F = \int_0^{2\pi}\int_0^{\pi/2} B(T)\cos\theta\sin\theta\, d\theta\, d\phi$$

$$= B(T)\int_0^{2\pi}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\, d\phi = \pi B(T) = \frac{ac}{4}T^4 \equiv \sigma T^4$$

This is Stefan-Boltzmann law

- From the derivation by thermodynamics, the Stefan-Boltzmann constant is just an integration constant, which has to be determined by experiment. However, it is in fact a composition of the fundamental physics constants (c, k, h), which will be shown in the later of this chapter.

3-2-2 Energy Distribution of the Radiation in a Vacuum Box

- In this subsection, we will use frequency instead of wavelength for discussion.
- The transformation between frequency to wavelength is simple, just use the following fact:

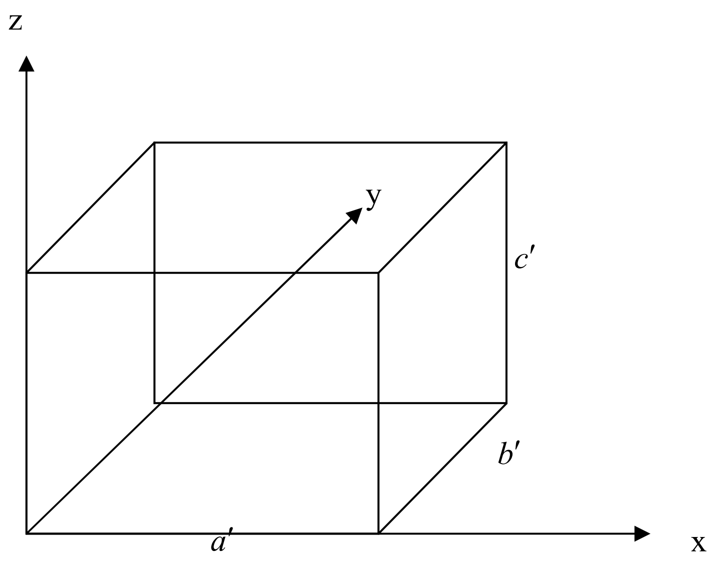
$$\nu = \frac{c}{\lambda} \Rightarrow d\nu = -\frac{c}{\lambda^2}$$

Thus any function $f(\nu)$

$$f_\nu(\nu)\,d\nu = f_\nu(\nu)\left|\frac{d\nu}{d\lambda}\right|d\lambda = f_\nu(\nu)\frac{c}{\lambda^2}d\lambda = f_\nu\left(\frac{c}{\lambda}\right)\frac{c}{\lambda^2}d\lambda = f_\lambda(\lambda)\,d\lambda$$

$$\therefore f_\lambda(\lambda) = f_\nu\left(\frac{c}{\lambda}\right)\frac{c}{\lambda^2}$$

- Since the B_ν, as well as u_ν is a universal function of temperature, we discuss the radiation (electromagnetic wave)in a box.



- Consider a box of size a, b and c. The **electromagnetic wave** inside the box is satisfied the wave equation

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t^2}$$

where $c =$ speed of light

- And **boundary condition**

$$\vec{E}(boundary) = 0$$

- **The monochromic solution** can be written as

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

where $\omega = 2\pi\nu$

- Substitute the general solution into wave equation

$$\left|\vec{k}\right|^2 \vec{E} = \frac{\omega^2}{c_l^2}\vec{E}$$

$$\left|\vec{k}\right|^2 = \frac{\omega^2}{c_l^2} = \frac{4\pi^2\nu^2}{c_l^2} = \frac{4\pi^2}{\lambda^2}$$

$$k = \left|\vec{k}\right| = \frac{2\pi}{\lambda}$$

$\vec{k}$: wave vector

- Alternatively, let

$$\vec{E}(\vec{r},t) = e^{-i\omega t}\vec{\varepsilon}(\vec{r})$$

- From wave equation

$$\nabla^2 \bar{\varepsilon} + k^2 \bar{\varepsilon} = 0$$

Let

$$\bar{\varepsilon}(x, y, z) = \bar{\varepsilon}_0 X(x) Y(y) Z(z)$$

$$\Rightarrow \frac{d^2 X}{dx^2} YZ + X \frac{d^2 Y}{dy^2} Z + XY \frac{d^2 Z}{dz^2} + k^2 XYZ = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = -k^2$$

- Since the three terms on the left-hand side of the equation depend on x, y, z only respectively, the equation can be divided into

$$\begin{cases} \frac{1}{X}\frac{d^2 X}{dx^2} = -k_x^2 \\ \frac{1}{Y}\frac{d^2 Y}{dy^2} = -k_y^2 \\ \frac{1}{Z}\frac{d^2 Z}{dz^2} = -k_z^2 \end{cases}$$

where

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

- Taking the **x direction** as example

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -k_x^2$$

$$\frac{d^2 X}{dx^2} + k_x^2 X = 0$$

General solution of X can be written as

$$X(x) = A_x \sin(k_x x + \phi_x)$$

- Consider the **boundary condition**

$$X(0) = 0$$

$$\Rightarrow X(x) = A_x \sin(k_x x)$$

$$X(a') = 0$$

$$\Rightarrow X(a) = A_x \sin(k_x a')$$

$$\Rightarrow k_x a' = n_x \pi$$

where $n_x = 0, 1, 2 \cdots\cdots$

$$\therefore k_x = \frac{n_x \pi}{a'}$$

$$X(x) = A_x \sin\left(\frac{n_x \pi}{a'} x\right)$$

- **Similarly**

$$Y(x) = A_y \sin\left(\frac{n_y \pi}{b'} y\right)$$

$$Z(z) = A_z \sin\left(\frac{n_z \pi}{c'} z\right)$$

- Thus

$$\vec{\varepsilon} = \vec{\varepsilon}_0 \sin\left(\frac{n_x\pi}{a'}x\right)\sin\left(\frac{n_y\pi}{b'}y\right)\sin\left(\frac{n_z\pi}{c'}z\right)$$

$$k^2 = \left(\frac{n_x\pi}{a'}\right)^2 + \left(\frac{n_y\pi}{b'}\right)^2 + \left(\frac{n_z\pi}{c'}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2 = \left(\frac{2\pi\nu}{c}\right)^2$$

- **Thus, only the wave vector satisfied equation above can exist in the box.**

- **Number of states**
  - Consider a space form by (n_x, n_y, n_z), averagely speaking each state occupies a unit volume. Thus,

$$N_{st} = n_x n_y n_z = \frac{k_x a'}{\pi}\frac{k_y b'}{\pi}\frac{k_z c'}{\pi}$$

$$= \frac{V}{\pi^3} k_x k_y k_z$$

  - Similar to the real space, the number of state for $k < k_0$

$$N_{st}(k < k_0) = \frac{1}{8}\frac{4\pi}{3}\frac{V}{\pi^3}k_0^3 = \frac{1}{6}\frac{V}{\pi^2}k_0^3$$

  - Note that since $n_x > 0; n_y > 0; n_z > 0$, we take only positive eighth of the ellipsoid.
  - For **number of state between** $k \to k + dk$

$$dN_{st} = \frac{1}{8}\frac{V}{\pi^3}4\pi k^2 dk = \frac{1}{2}\frac{V}{\pi^2}k^2 dk$$

Since $k = \frac{2\pi\nu}{c}$

$$dN_{st} = \frac{1}{2}\frac{V}{\pi^2}\left(\frac{2\pi}{c}\right)^3 \nu^2 d\nu = \frac{4\pi V}{c^3}\nu^2 d\nu$$

- **The each component of electromagnetic wave (i.e. each (n_x, n_y, n_z)) can be treated as a simple harmonic oscillator freely vibrating.**

- **The Hamiltonian of a simple harmonic oscillator can be written as**

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

- **Using the equipartition theorem, the mean energy of each square term is $1/2\,kT$.** Thus,

$$\langle \varepsilon \rangle = 2 \cdot \frac{1}{2}kT = kT$$

- For the electromagnetic wave, there are **two polarizations**, so

$$\langle \varepsilon \rangle = 2kT$$

- Thus, total energy between $\nu \to \nu + d\nu$ inside the box is

$$dU = \langle \varepsilon \rangle \frac{4\pi V}{c^3} \nu^2 d\nu = \frac{8\pi V k T}{c^3} \nu^2 d\nu$$

$$u_\nu = \frac{dU}{V d\nu} = \frac{8\pi k T}{c^3} \nu^2$$

or express in wavelength λ

$$u_\lambda = \frac{dU}{V d\lambda} = \frac{dU}{V d\nu} \left| \frac{d\nu}{d\lambda} \right| = \frac{8\pi k T}{c^3} \left(\frac{c}{\lambda} \right)^2 \frac{c}{\lambda^2} = \frac{8\pi k T}{\lambda^4}$$

- Since the black body is isotropic, from the equation in Chapter 2

$$I_\nu = J_\nu = \frac{c}{4\pi} u_\nu = \frac{2kT}{c^2} \nu^2$$

- The equation above is call **Rayleigh-Jeans law**, **which agrees well for the low frequency part of black body** radiation but **not in high frequency part.**

- However, the energy per unit volume:

$$u = \int_0^\infty \frac{8\pi k T}{c^3} \nu^2 d\nu = \frac{8\pi k T}{c^3} \int_0^\infty \nu^2 d\nu \to \infty$$

- It is called ultraviolet catastrophe. Clearly it is wrong.

2-2-3 Planck Spectrum

- **Max Planck suggested that the energy of the electromagnetic wave is not continuous but discrete (called photon)**

$$E_\nu = nh\nu$$

- **Use the Boltzmann distribution:**

Let $\beta = \frac{1}{kT}$; $\quad E = h\nu$

$$\langle \varepsilon_\nu \rangle = 2 \frac{\sum_{n=1}^{\infty} nE e^{-n\beta E}}{\sum_{n=1}^{\infty} e^{-n\beta E}} = 2 \frac{1}{\sum_{n=1}^{\infty} e^{-n\beta E}} \left(-\frac{\partial}{\partial \beta} \sum_{n=1}^{\infty} e^{-n\beta E} \right)$$

$$= -2 \frac{\partial}{\partial \beta} \ln \left(\sum_{n=1}^{\infty} e^{-n\beta E} \right)$$

$$\sum_{n=1}^{\infty} e^{-n\beta E} = \frac{1}{1 - e^{-\beta E}}$$

$$\frac{\partial}{\partial \beta} \sum_{n=1}^{\infty} e^{-n\beta E} = \frac{\partial}{\partial \beta} \frac{1}{1 - e^{-\beta E}} = -\left(\frac{1}{1 - e^{-\beta E}} \right)^2 E e^{-\beta E}$$

$$\therefore \langle \varepsilon_\nu \rangle = 2 \frac{\left(\frac{1}{1 - e^{-\beta E}} \right)^2 E e^{-\beta E}}{\frac{1}{1 - e^{-\beta E}}} = \frac{2 E e^{-\beta E}}{1 - e^{-\beta E}} = \frac{2E}{e^{\beta E} - 1} = \frac{2h\nu}{\exp\left(h\nu / kT \right) - 1}$$

- Thus,

$$dU = \langle \varepsilon \rangle \frac{4\pi V}{c^3} \nu^2 d\nu = \frac{h\nu}{\exp\left(h\nu/kT\right)-1} \frac{8\pi V}{c^3} \nu^2 d\nu$$

$$u_\nu = \frac{dU}{V d\nu} = \frac{8\pi h \nu^3}{c^3 \left[\exp\left(h\nu/kT\right)-1\right]}$$

or express in wavelength

$$u_\lambda = \frac{dU}{V d\lambda} = \frac{dU}{V d\nu}\left|\frac{d\nu}{d\lambda}\right| = \frac{8\pi hc}{\lambda^5 \left[\exp\left(hc_l/\lambda kT\right)-1\right]}$$

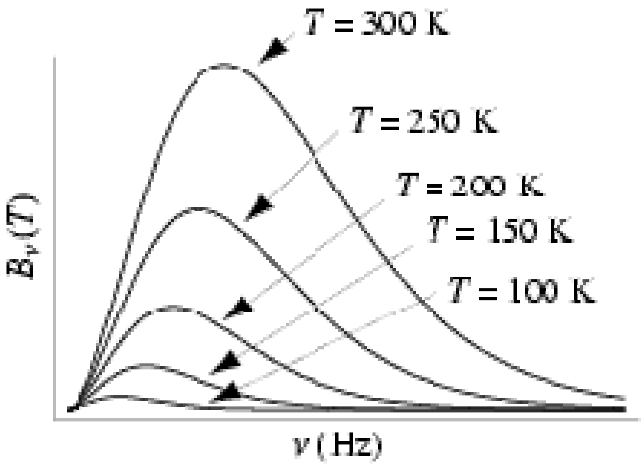
- **And the emission spectrum**

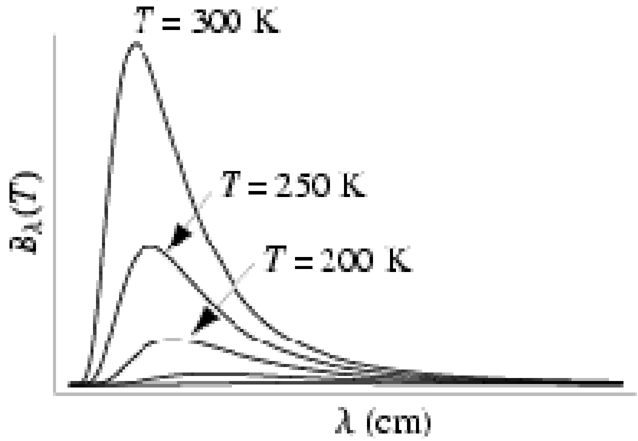
$$I_\nu = J_\nu = B_\nu = \frac{c}{4\pi} u_\nu = \frac{2h\nu^3}{c^2 \left[\exp\left(h\nu/kT\right)-1\right]}$$

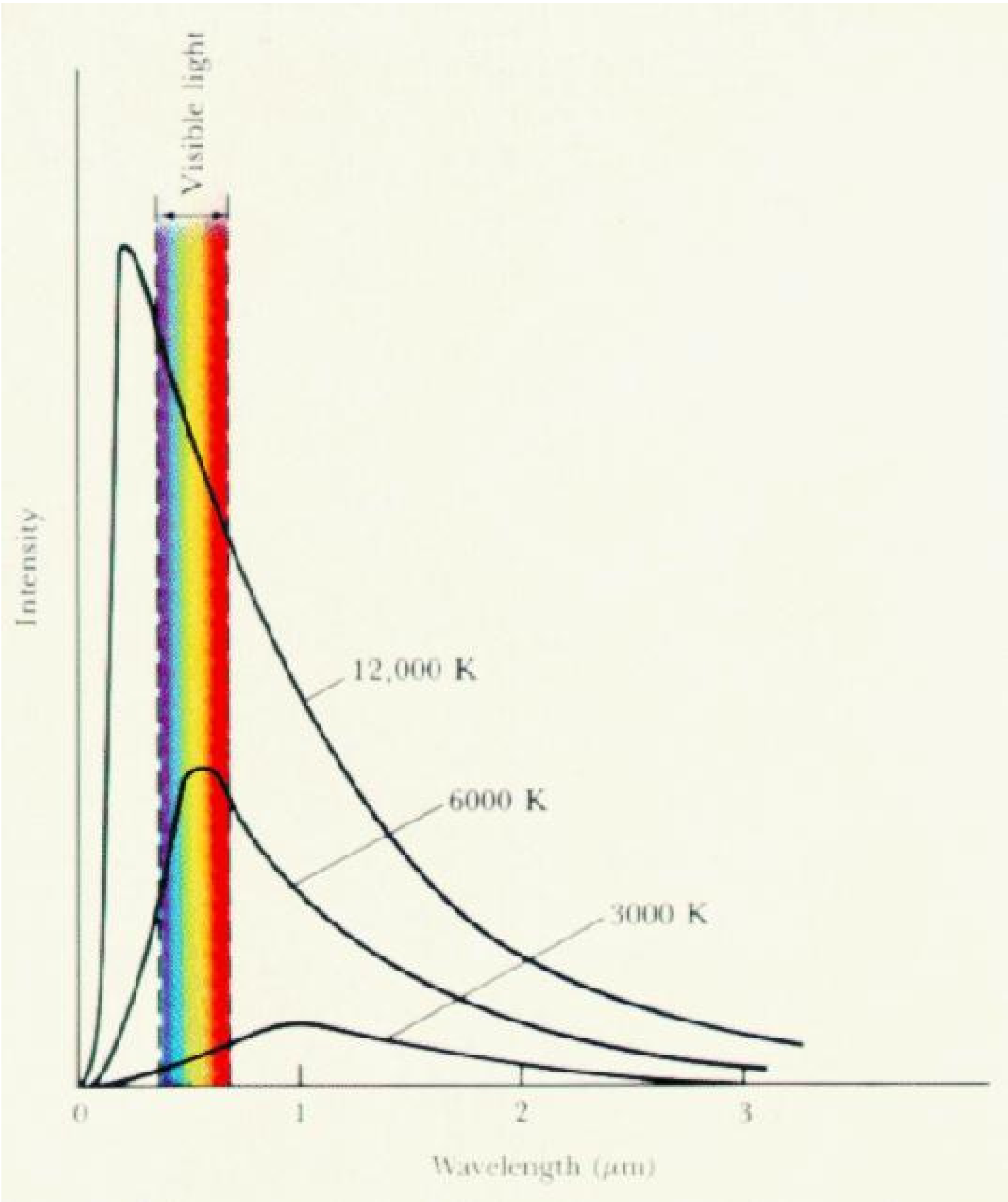
or express in λ

$$I_\lambda = J_\lambda = B_\lambda = \frac{c}{4\pi} u_\lambda = \frac{2hc^2}{\lambda^5 \left[\exp\left(hc_l/\lambda kT\right)-1\right]}$$

That is block body radiation




T = 300 K
T = 250 K
T = 200 K
$B_\lambda(T)$
λ (cm)


Visible light
Intensity
12,000 K
6000 K
3000 K
0
1
2
3
Wavelength (μm)

3-3 Properties of Planck Law

3-3-1 Stephen-Boltzmann Law

- Integration over all the frequencies

$$u_\nu(T) = \frac{8\pi h\nu^3}{c^3\left[\exp\left(h\nu/kT\right)-1\right]}$$

$$u(T) = \int_0^\infty \frac{8\pi h\nu^3}{c^3\left[\exp\left(h\nu/kT\right)-1\right]} d\nu$$

Let $x = h\nu/kT \Rightarrow \nu = xkT/h$

$$u(T) = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x-1} dx$$

$$\int_0^\infty \frac{x^3}{e^x-1} dx = \int_0^\infty \frac{x^3 e^{-x}}{1-e^{-x}} dx = \int_0^\infty x^3 \sum_{n=1}^{\infty} e^{-nx} dx$$

$$= \sum_{n=1}^{\infty} \int_0^\infty x^3 e^{-nx} dx = \sum_{n=1}^{\infty} \frac{1}{n^4} \int_0^\infty (nx)^3 e^{-(nx)} d(nx)$$

$$= 6\sum_{n=1}^{\infty} \frac{1}{n^4}$$

Riemann zeta function of 4

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(4) = \frac{\pi^4}{90}$$

$$\int_0^\infty \frac{x^3}{e^x-1} dx = \frac{\pi^4}{15}$$

$$\therefore u(T) = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \frac{\pi^4}{15} = \frac{8\pi^5 k^4}{15c^3h^3} T^4 = aT^4$$

- **Thus Stephen-Boltzmann constants are**

$$a = \frac{8\pi^5 k^4}{15c^3h^3} = 7.56\times10^{-15}\ \text{erg cm}^{-3}\ \text{K}^{-4}$$

$$\sigma = \frac{ca}{4} = \frac{2\pi^5 k^4}{15c^2h^3} = 5.67051\times10^{-5}\ \text{erg cm}^{-2}\ \text{K}^{-4}\ \text{s}^{-1}$$

- The constant can be composed by the fundamental physical constants (c, k, h) from Planck distribution rather than an integration constant from thermodynamics.

3-3-2 Wien Displacement Law

- **The frequency** $\nu_{\max}$ **at which the peak of** $B_\nu(T)$ occurs can be found by solving

$$\left.\frac{\partial B_\nu}{\partial \nu}\right|_{\nu=\nu_{\max}} = 0$$

$$\frac{\partial}{\partial \nu}\left\{\frac{2h\nu^3}{c^2\left[\exp\left(h\nu/kT\right)-1\right]}\right\}=0$$

Let $x = h\nu/kT$

$$\Rightarrow \frac{d}{dx}\frac{x^3}{e^x-1}=\frac{3x^2\left(e^x-1\right)-x^3e^x}{\left(e^x-1\right)^2}=0$$

$$\Rightarrow x = 3\left(1-e^{-x}\right)$$

The numerical solution above is $x = 2.82$

$$\therefore h\nu_{\max} = 2.82kT$$

or

$$\frac{\nu_{\max}}{T} = 5.88\times10^{10}\,\text{Hz K}^{-1}$$

- Thus, the peak frequency of the black body law shifts linearly with temperature. This is known **Wien displacement law**.

- Alternative, frequency $\lambda_{\max}$ at which the peak of $B_\lambda(T)$ occurs can be found by solving

$$\left.\frac{\partial B_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_{\max}} = 0$$

$$\frac{\partial}{\partial \lambda}\left\{\frac{2hc^2}{\lambda^5\left[\exp\left(hc_l/\lambda kT\right)-1\right]}\right\}=0$$

Let $x = h/\lambda kT$

$$\frac{\partial}{\partial\lambda}=\frac{\partial x}{\partial\lambda}\frac{\partial}{\partial x}=-\frac{h}{\lambda^2 kT}\frac{\partial}{\partial x}$$

$$\Rightarrow \frac{d}{dx}\frac{x^5}{e^x-1}=\frac{5x^4\left(e^x-1\right)-x^5e^x}{\left(e^x-1\right)^2}=0$$

$$\Rightarrow x = 5\left(1-e^{-x}\right)$$

The numerical solution above is $x = 4.97$

$$\therefore \lambda_{\max}T = \frac{h}{4.97k}$$

or

$$\lambda_{\max}T = 0.290\text{ cm K}$$

- This is also known as **Wien displacement law.**

- Note that the peaks of B_ν and B_λ are different. For example, for T=7300 K, the peak of B_ν is at $\lambda = 7000\,\text{Å}$ but B_λ is at $\lambda = 4000\,\text{Å}$

- Roughly speaking, the *order* of peak frequency (wavelength) $h\nu_{peak} \approx \dfrac{hc}{\lambda_{peak}} \approx kT$

Wavelength	Frequency (Hz)	Photon energy (eV)	Temperature (K)	Wave band
1 cm	**$3.000\text{x}10^{10}$=30.00GHz**	1.24 $\text{x}10^{-4}$	1.43877826	Radio
5.27 mm	$5.688\text{x}10^{10}$=56.88GHz	2.3525 $\text{x}10^{-4}$	**2.73**	Radio
1 mm	**$3.000\text{x}10^{11}$=300.0GHz**	1.24 $\text{x}10^{-3}$	14.3877826	Radio→Sub mm
47.96 μm	$6.250\text{x}10^{12}$	0.026	**300.0**	Infrared
700 nm	4.283 x 10^{14}	1.77	20553.9751	Visible light
400 nm	7.495 x 10^{14}	3.09961072	35969.4565	Visible light
1.24 nm	2.418 x 10^{17}	**$1\text{x}10^3$=1 keV**	1.16 $\text{x}10^7$	X-ray
1.24 pm	2.418 x 10^{20}	**$1\text{x}10^6$=1 MeV**	1.16 $\text{x}10^{10}$	γ -ray

~1 °K	Radio
~10 °K	Sub mm
~10^2 °K	Far infrared
~10^3 °K	Near infrared
~10^4 °K	Visible
~10^5 °K	Ultraviolet
~10^6 °K	Ectreme Ultraviolet
~10^7 °K	Soft X-ray
~10^8 °K	Hard X-ray (rarely used)
~10^9 °K	Soft γ-ray (rarely used)
>10^{10} °K	γ-ray (rarely used)

3-3-3 Monotonicity with Temperature

- **For any given frequency ν if and only if $T_2 > T_1$ then $B_\nu(T_2) > B_\nu(T_1)$**

- Proof

If the statement above is true

$$\Rightarrow B_\nu(T + dT) - B_\nu(T) > 0 \text{ if } dT > 0 \text{ and vise versa}$$

$$\Rightarrow \frac{B_\nu(T + dT) - B_\nu(T)}{dT} = \frac{\partial B_\nu(T)}{\partial T} > 0$$

$$\frac{\partial B_\nu(T)}{\partial T} = \frac{2h\nu^4}{c^2 kT^2} \frac{\exp(h\nu/kT)}{\left[\exp(h\nu/kT) - 1\right]^2} > 0$$

- Also note that $B_\nu \to 0$ as $T \to 0$ and $B_\nu \to \infty$ as $T \to \infty$

3-3-4 Rayliegh-Jeans Law

- Looking at the **low frequency spectrum**, $h\nu << kT$

$$\frac{h\nu}{kT} << 1$$

$$\exp\left(\frac{h\nu}{kT}\right) - 1 \approx \frac{h\nu}{kT}$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2\left[\exp(h\nu/kT) - 1\right]}$$

$$\approx \frac{2h\nu^3}{c^2 \dfrac{h\nu}{kT}} = \frac{2kT^2}{c^2}\nu^2$$

- This is **Rayliegh-Jeans law.**

- Note that the equation above has no Planck constant in it. That is why it can be derived before the birth of quantum mechanics.

- Rayliegh-Jeans law is considering the radiation as wave instead of photons. This also indicates that the radiation act like a wave rather than particle for low frequency.

- **Rayliegh-Jeans law applies at low frequency. In the radio region it almost always applies.**

3-3-5 Wien Law

- **Looking at the high frequency spectrum,** $h\nu >> kT$

$$\frac{h\nu}{kT} >> 1$$

$$\exp\left(\frac{h\nu}{kT}\right) >> 1$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2\left[\exp(h\nu/kT) - 1\right]}$$

$$\approx \frac{2h\nu^3}{c^2 \exp(h\nu/kT)} = \frac{2h\nu^3}{c^2}\exp(-h\nu/kT)$$

- **This is the Wein law.**

- The brightness decreases rapidly as the temperature increase.

3-4 Characteristic Temperature Related to Planck Spectrum

3-4-1 Brightness Temperature

- One way to characterizing brightness (specific intensity I_ν) at a certain frequency is to give the temperature of black body having the same brightness at that frequency.

$$I_\nu = B_\nu\left(T_b\right)$$

- T_b : brightness temperature

- **The brightness temperature is frequently used in radio astronomy, where the Rayleigh-Jeans law is usually applicable and thus**,

$$I_\nu = \frac{2\nu^2}{c^2} kT_b$$

$$\Rightarrow T_b = \frac{c^2}{2\nu^2 k} I_\nu$$

- Therefore, in the Rayleigh-Jeans limit ($h\nu << kT$), from the radiation transfer equation

$$T_b = T_b\left(0\right)e^{-\tau_\nu} + T\left(1-e^{-\tau_\nu}\right)$$

where T is the temperature of the atmosphere and

T_b is the (observed) brightness temperature

- For large optical depth, $T = T_b$

3-4-2 Color Temperature

- **The color temperature T_c is obtained by measuring the shape spectrum**

- The observed spectrum is fitted to a black body spectrum with a free parameter T.

- In practical application, we seldom care about the vertical scale in the spectrum but compare the shape only.

3-4-3 Effective Temperature

- **The effect temperature T_{eff} is measured by the Stephen-Boltzmann law**

$$F_s = \sigma T_{eff}^4$$

where F_s is the flux of the star's surface

- If the emission is spherical symmetry

$$L = 4\pi R^2 \sigma T_{eff}^4 = 4\pi d^2 f$$

R : radius of star

d : distance from the star to Earth

f : measured (bolometric) flux

- Note: Both T_{eff} and T_b depend on the magnitude of the source so the distance must be known in advance, but T_c is depend on the shape of the spectrum and the distance is not required.

3-5 Local Thermodynamic Equilibrium

3-5-1 What is Local Thermodynamic Equilibrium?

- All the thermodynamic quantities ($U, S, p, T \cdots$) are macroscopic and some of them, such as p and T (the intensive parameters) can only defined in the equilibrium state. Than is, these values are same over all system.

- However, the stellar atmosphere can not be in the ideal thermodynamic equilibrium state as described above. For example, there is temperature gradient in the atmosphere (hotter inside and cooler outside).

- However, we can treat (or assume) that the stellar atmosphere is in Local Thermodynamic Equilibrium (LTE).

- **LTE: the thermodynamic equilibrium is valid for an infinitesimal but still macroscopic region in the stellar atmosphere.**

3-5-2 Is the LTE a Good Assumption?

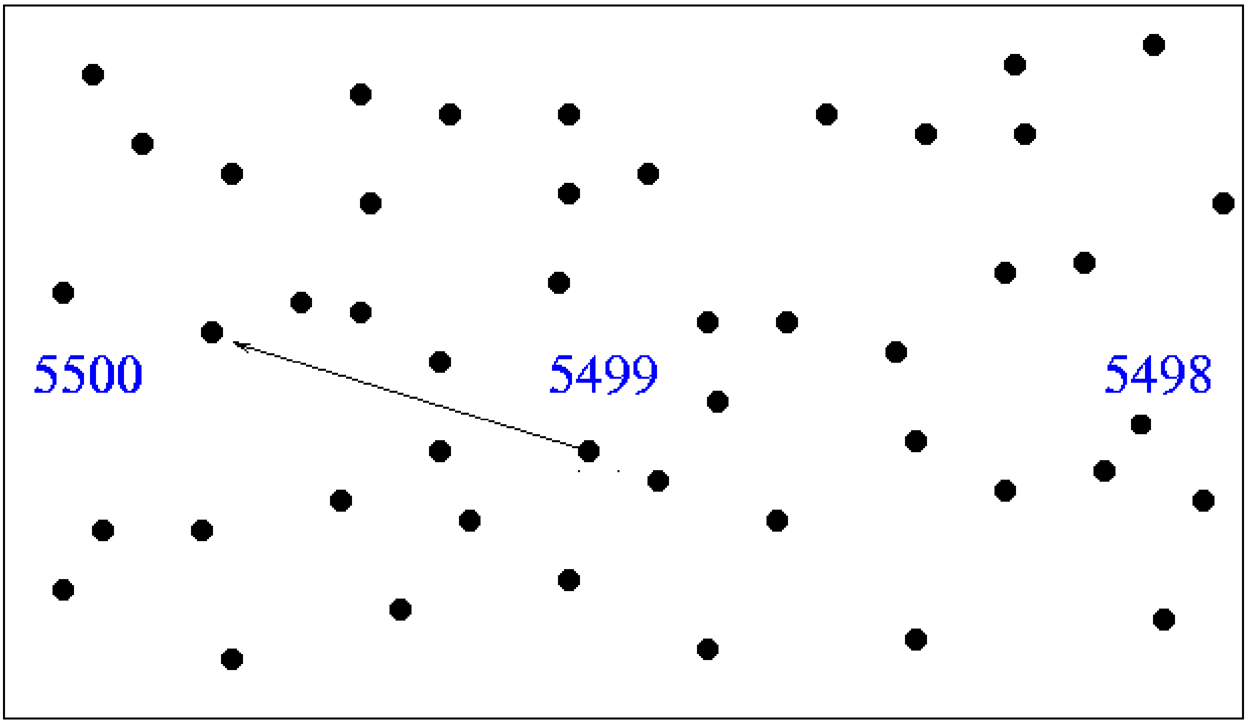
- Clearly, the stellar atmosphere is not exact LTE, but if it is a good assumption?

- **The LTE is a good assumption if the mean free path of the particle is much less than the scale length of the change of the temperature.**

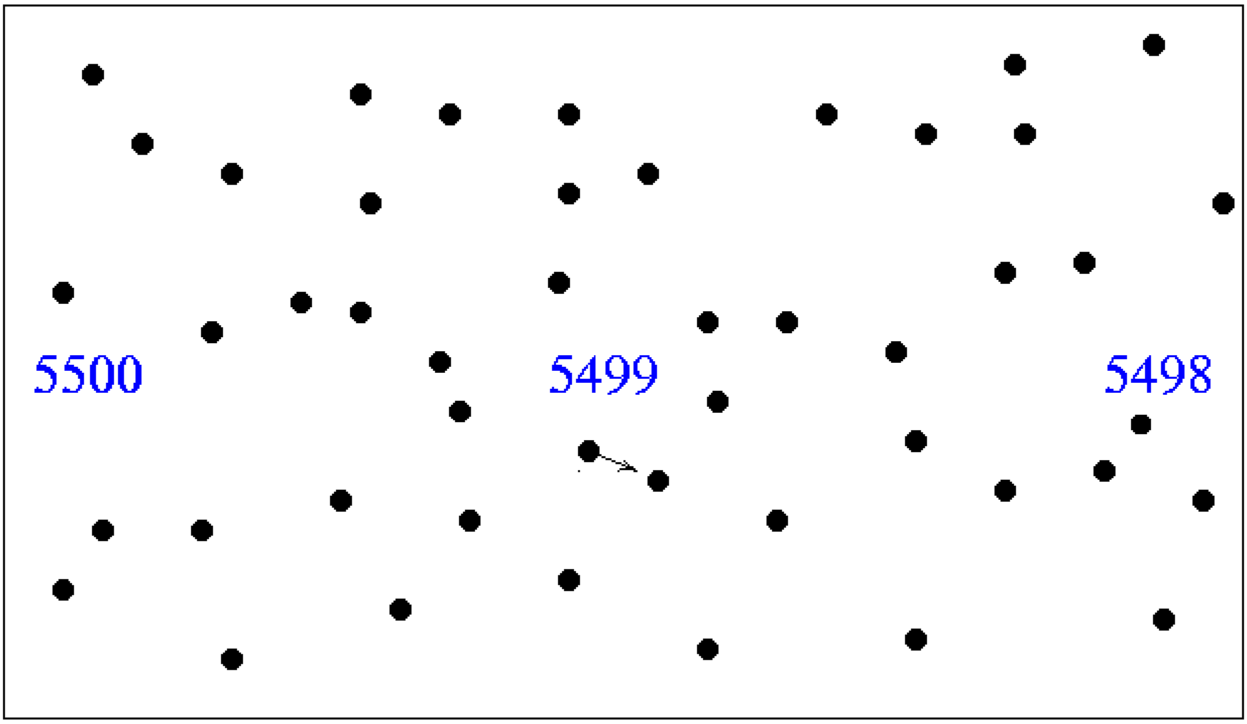
- **The scale length of the change of the temperature: the distance over which some property changes "significantly" as the ratio of the *local* value to the *local gradient*.** For example the temperature scale height can be define as

$$H = \frac{T}{\frac{dT}{ds}}$$

- **If the gas particle can be easily traveling through the scale length,** then it will constantly be changing its properties with every collision. That means that **we will never be able to describe it for any length of time as having some well defined temperature and the LTE assumption fail**



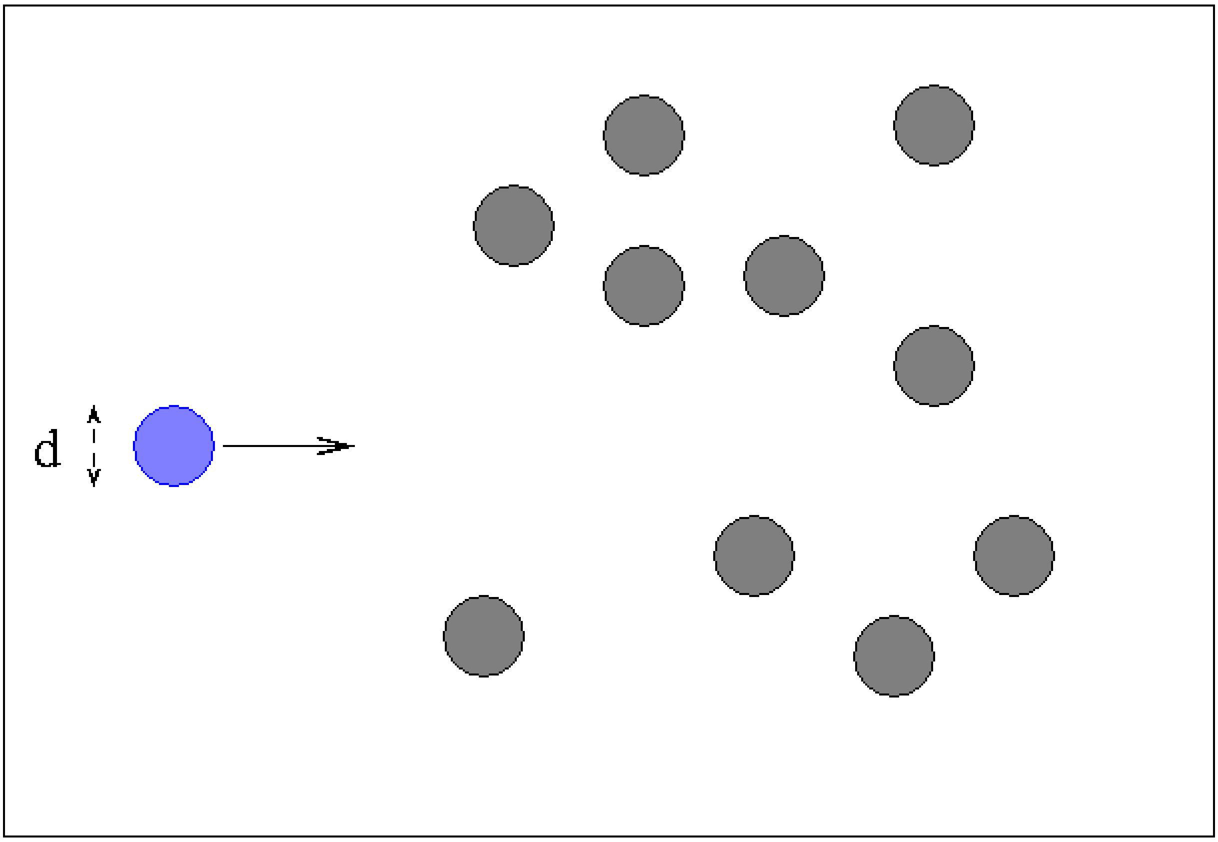
- **On the other hand, if the mean free path of the particle is very small compared to the distance over which the temperature changes**, then the particle will collide many times with other atoms, all of the same temperature, before it can possibly reach some region with a different temperature. In this case, the speed of particles within some small region may very well be described by a Maxwell-Boltzmann distribution with a definite temperature.



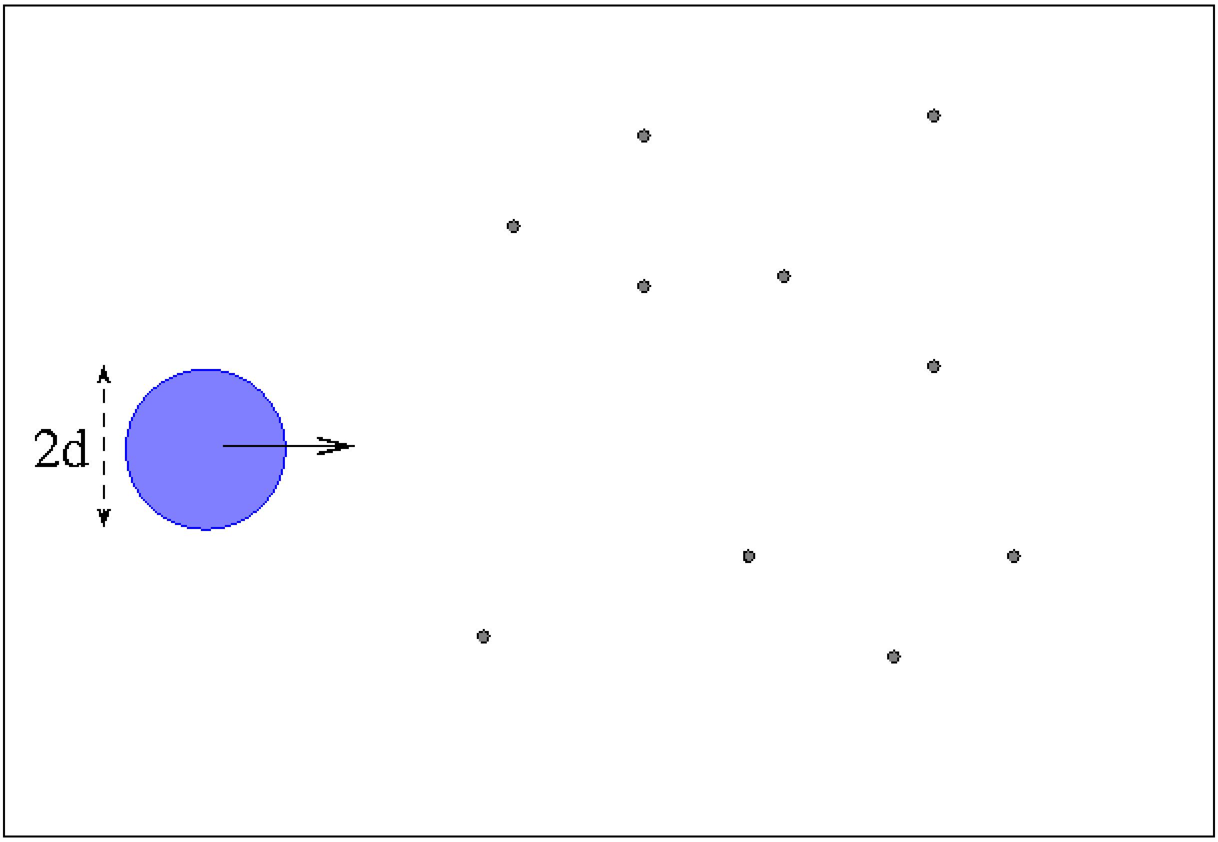
3-5-2 Mean Free Path of the Particle of the Gas

- **Consider the particle of the gas is like a hard sphere with radius r and the diameter $d = 2r$.**

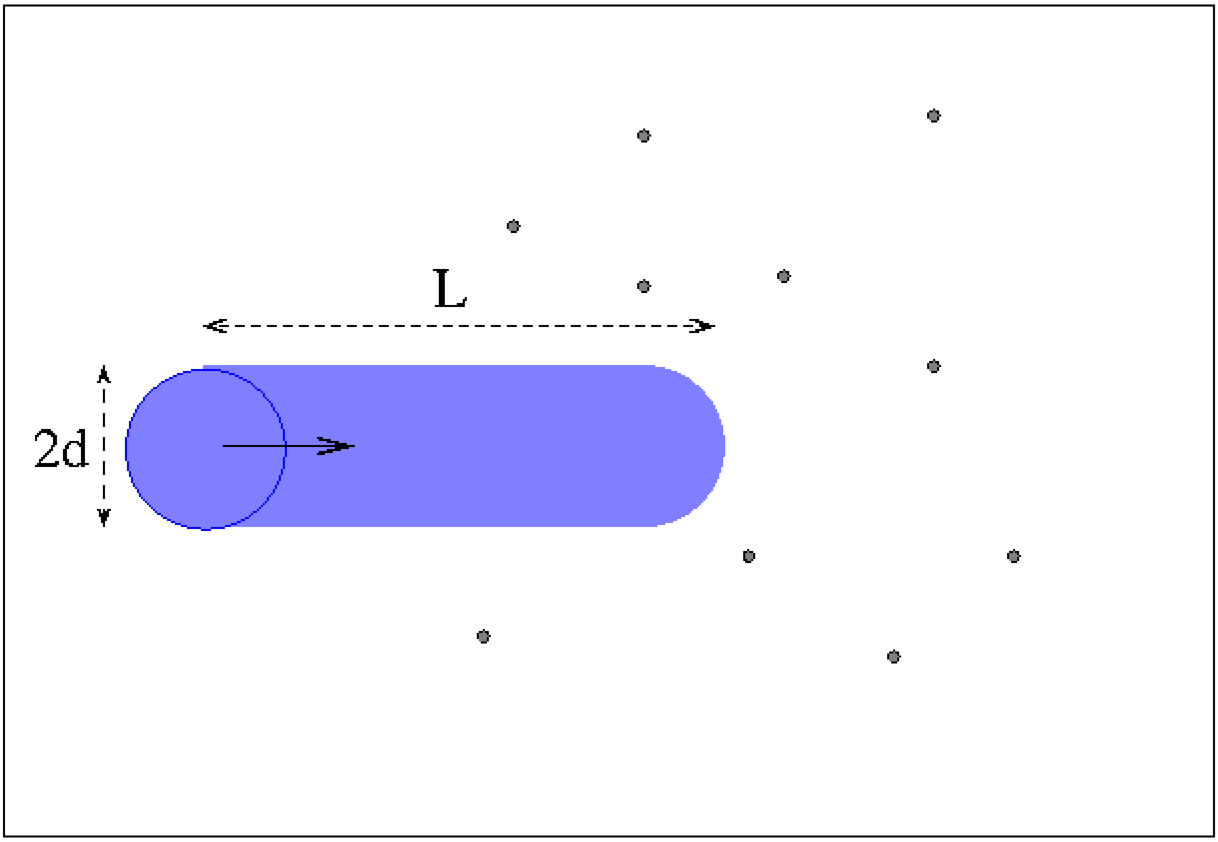
- For a specified particle pass through the gas with number density n, since the gas is in equilibrium state, number of particles in a certain volume is a constant. We thus can consider the particles except the specified one are static.



- To calculate the collision frequency of the specified particle, we can make the specified particle's radius $r \rightarrow 2r$ and the other ones have zero size.



- Now, from $t \to t + \Delta t$ as the specified particle moves through the sea of other particles, it will sweep out a cylindrical path.



- All the other particles inside the cylinder would collide with the specified particle, therefore,

Cylinder volume $V = \pi (2r)^2 \cdot L = 4\pi r^2 \cdot v\Delta t$

where is v speed of the specified particle.

The number of particles inside the cylinder is

$nV = 4\pi r^2 n \cdot v\Delta t$

All the particles inside the cylinder would collide the specified one between $t \to t + \Delta t$

$\therefore$ Nummber of collisions between $t \to t + \Delta t$ is $nV = 4\pi r^2 n \cdot v\Delta t$

Path length : $v\Delta t$

Mean free path: the mean path length between two consecutive collisions

$l = \dfrac{v\Delta t}{nV} = \dfrac{v\Delta t}{4\pi r^2 n \cdot v\Delta t} = \dfrac{1}{4\pi r^2 n} = \dfrac{1}{n\sigma}$

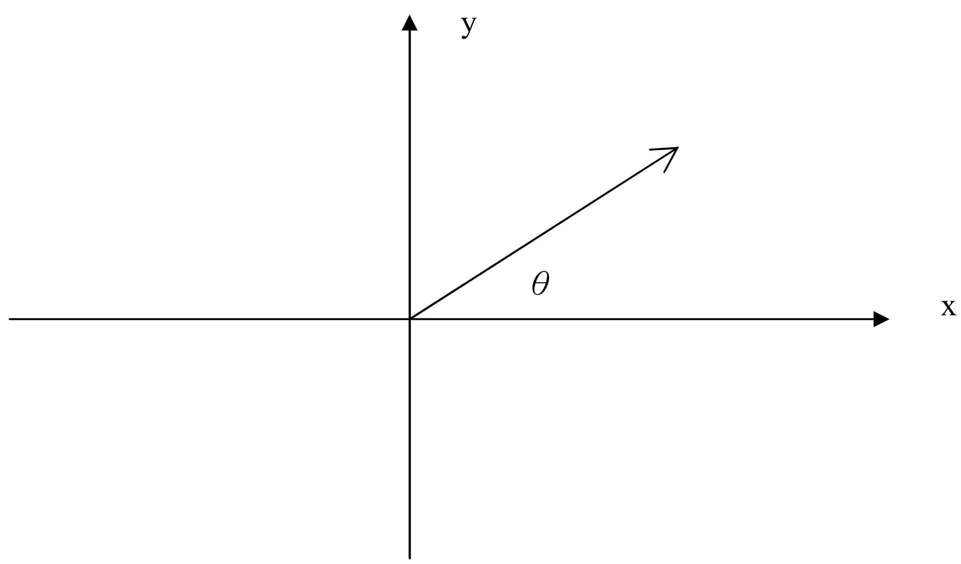
where is the collision cross section of the particle.

In this case, $\sigma = 4\pi r^2 = \pi d^2$

- Thus, larger particle density and/or larger collision cross section give smaller mean free path.
- Mean collision frequency is

$$f_c = \frac{v}{l} = nv\sigma$$

- **In fact, every collision makes the particle change direction.** Since the collision is isotropic, the scattered direction can be considered as uniform randomly distributed over all the direction, thus, like the **random walks.**
- Taking 2-D random walks as example:



- **Since the direction random, the probability that the particle goes into angle $\theta \to \theta+d\theta$ is**

$$P(\theta)d\theta = \frac{d\theta}{2\pi}$$

- After N steps, the final position of particle

$$\vec{R} = \sum_{i=1}^{N} \vec{r}_i = X\hat{x} + Y\hat{y}$$

$$\vec{r} = x_i\hat{x} + y_i\hat{y}$$

$$x_i = l\cos\theta_i$$

$$y_i = l\sin\theta_i$$

$$X = \sum_{i=1}^{N} x_i$$

$$Y = \sum_{i=1}^{N} y_i$$

- **The mean position of the particle**

$$\langle X \rangle = \left\langle \sum_{i=1}^{N} x_i \right\rangle = \sum_{i=1}^{N} \langle x_i \rangle$$

$$\langle x_i \rangle = \int_0^{2\pi} x_i P(\theta_i)d\theta_i = \frac{l}{2\pi}\int_0^{2\pi} \cos\theta_i d\theta_i = 0$$

$$\langle Y \rangle = \left\langle \sum_{i=1}^{N} y_i \right\rangle = \sum_{i=1}^{N} \langle y_i \rangle$$

$$\langle y_i \rangle = \int_0^{2\pi} y_i P(\theta_i)d\theta_i = \frac{l}{2\pi}\int_0^{2\pi} \sin\theta_i d\theta_i = 0$$

- **Thus, the mean position is** $\langle \vec{R} \rangle = \langle X \rangle \hat{x} + \langle Y \rangle \hat{y} = 0$
- **However, the "deviation" from the mean**

$$\langle S_x^2 \rangle = \left\langle \sum_{i=1}^{N} x_i^2 \right\rangle = \sum_{i=1}^{N} \langle x_i^2 \rangle$$

$$\langle x_i^2 \rangle = \int_0^{2\pi} x_i^2 P(\theta_i)d\theta_i = \frac{l^2}{2\pi}\int_0^{2\pi} \cos^2\theta_i d\theta_i = \frac{l^2}{2\pi}\int_0^{2\pi} \frac{1+\cos 2\theta_i}{2} d\theta_i = \frac{l^2}{2\pi}\pi = \frac{l^2}{2}$$

$$\therefore \langle S_x^2 \rangle = N\frac{l^2}{2}$$

$$\langle S_y^2 \rangle = \left\langle \sum_{i=1}^{N} y_i^2 \right\rangle = \sum_{i=1}^{N} \langle y_i^2 \rangle$$

$$\langle y_i \rangle = \int_0^{2\pi} y_i^2 P(\theta_i)d\theta_i = \frac{l^2}{2\pi}\int_0^{2\pi} \sin^2\theta_i d\theta_i = \frac{l^2}{2\pi}\int_0^{2\pi} \frac{1-\cos 2\theta_i}{2} d\theta_i = \frac{l^2}{2\pi}\pi = \frac{l^2}{2}$$

$$\therefore \langle S_y^2 \rangle = N\frac{l^2}{2}$$

- Mean deviation

$$R = \sqrt{\langle S_x^2 \rangle + \langle S_y^2 \rangle} = l\sqrt{N}$$

- The "diffusion" :

$$R = l\sqrt{N} = l\sqrt{f_c t} = l\sqrt{\frac{v}{l}t} = \sqrt{vlt}$$

- For sufficient small mean free path and large temperature length scale, it would take long time to diffuse out of a temperature length scale.
- Similar argument is also true for radiation.

-

3-5-3 What if the LTE Valid or Invalid?

- **Electrons and ion velocity distributions are Maxwellian**

Speed distribution

$$f(v) = n\left[4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\right]v^2 \exp\left(-\frac{mv^2}{2kT}\right)$$

- **Excitation equilibrium is given by Boltzmann equation.**

- **Ionization equilibrium is given by Saha equation.**

- **The source function is given by the Planck function.**

- **Consequently, *LTE is a good assumption in stellar interiors since they have large density, but may break down in the atmosphere.***

- If LTE is no longer valid, all processes need to be calculated in detail via non-LTE. This is much more complicated, but needs to be considered in some cases